

Matone's relation of $\mathcal{N} = 2$ super Yang-Mills and spectrum of Toda chain

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Abstract

In $\mathcal{N} = 2$ super Yang-Mills theory, the Matone's relation relates instanton corrections of the prepotential to instanton corrections of scalar field condensation $\langle \text{Tr} \varphi^2 \rangle$. This relation has been proved to hold for Omega deformed theories too, using localization method. In this paper, we first give a case study supporting the relation, which does not rely on the localization technique. Especially, we show that the magnetic expansion also satisfies a relation of Matone's type. Then we discuss implication of the relation for the spectrum of periodic Toda chain, in the context of recently proposed Nekrasov-Shatashvili scheme.

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1 Introduction

The $\mathcal{N} = 2$ super Yang-Mills theory can be analytically solved along the line of Seiberg-Witten theory of SU(2) gauge theory[1]. The low energy dynamics of Coulomb phase of gauge theory is encoded in the prepotential $\mathcal{F}(a)$, a holomorphic function of the complex quantity a . And $\frac{1}{2}a\sigma_3$ is the vacuum expectation value(v.e.v) of the complex scalar field φ , $u(a) = \langle \text{Tr}\varphi^2 \rangle$ is the proper parameter for the moduli space. The prepotential receives contributions from perturbative(tree and 1-loop) effects and nonperturbative effects,

$$\mathcal{F}(a) = \mathcal{F}^{pert}(a) + \mathcal{F}^{inst}(a). \quad (1)$$

They have the form of power expansion as

$$\mathcal{F}^{pert}(a) = \frac{i}{\pi}a^2(\ln(\frac{2a}{\Lambda}) - \frac{3}{2}), \quad \mathcal{F}^{inst}(a) = \sum_{k=1}^{\infty} \frac{i}{\pi}c_k a^2 (\frac{\Lambda}{a})^{4k}. \quad (2)$$

where Λ is the scale of gauge theory. $\mathcal{F}(a)$ and $u(a)$ satisfy the following remarkable relation,

$$2\mathcal{F}(a) - a \frac{\partial \mathcal{F}(a)}{\partial a} = \frac{2u}{i\pi}. \quad (3)$$

The most interesting, and nontrivial, part of the prepotential is the nonperturbative instanton part. Let us introduce instanton expansion parameter $q = \Lambda^4$ and expand $\mathcal{F}(a)$ and $u(a)$ in terms of k -instanton contributions,

$$-\mathcal{F}^{inst} = \sum_{k=1}^{\infty} \mathcal{F}_k q^k, \quad (4)$$

$$2u(a) = a^2 + \sum_{k=1}^{\infty} \mathcal{G}_k q^k. \quad (5)$$

Therefore we have $\mathcal{F}_k = (\frac{1}{i\pi})c_k a^{2-4k}$, and the equation (3) leads to the Matone's relation[2]²

$$\mathcal{G}_k = k\mathcal{F}_k. \quad (6)$$

Equation (3) and (6) are equivalent for Seiberg-Witten theory, however as we will see later, for Ω deformed theory equation (6) is preserved while (3) is not.

This relation was first found in [2] for SU(2) theory without matter by direct using Seiberg-Witten data. Generalization to higher rank gauge groups were carried out in [3]. In the instanton calculation frame work this relation was proved in [4] up to two instantons, in

²Our notations use a different normalization compared to [2]. We have absorbed a $4\pi i$ factor into \mathcal{F}_k , and \mathcal{G}_k is the expansion for $2u$ compared to that in [2] for u , therefore there is a $2\pi i$ difference compared to (14) of [2].

[5] for multi-instanton contributions. It was also proved in [6] through the superconformal Ward identities of super Yang-Mills applied to $\mathcal{N} = 2$ gauge theory with broken gauge symmetry.

In fact, the prepotential depends on the scale Λ in the form $\mathcal{F}(a, \Lambda) = a^2 f(\frac{\Lambda}{a})$, the left hand side of equation (3) can be interpreted as the renormalization flow of the prepotential,

$$2\mathcal{F}(a) - a \frac{\partial \mathcal{F}(a)}{\partial a} = \Lambda \frac{\partial \mathcal{F}(a)}{\partial \Lambda}. \quad (7)$$

The renormalization flow therefore incorporates the instanton effects. This renormalization interpretation is not only limited to the pure gauge theory[7], it was also derived in [8] for SU(N) theory with massive fundamental matter, in [9] for SU(N) theory with adjoint matter, and in [10] for general gauge groups with matter.

An even more interesting fact is that the Matone's relation is a result of integrability of Seiberg-Witten theory [11, 12, 13, 14]. In a simpler form, the integrability of Seiberg-Witten theory can be understood as a precise correspondence between solutions of Coulomb branch of various $\mathcal{N} = 2$ theories and certain classical integrable systems. For example, for the SU(N) pure $\mathcal{N} = 2$ gauge theory, the corresponding integrable system is the N-particle periodic Toda chain. On a deeper level, the $\mathcal{N} = 2$ theory/integrable models can be embedded into the Whitham-Toda theory of integrable hierarchy [11, 13, 14]. The Whitham-Toda system consists of a spectral curve equipped with a meromorphic differential dS depending on the modulated moduli $h_k(T_n)$ and infinite number of “slow” time variables $T_n, n \geq 1$ (compared to the “fast” time variables which govern the evolution of high frequency oscillations, after averaging these fast modes we get the Whitham dynamics of remaining slow modes [15, 16]). With these data we can construct period integrals of dS on the spectral curve, its τ -function, and a whole family of flow equations for τ -function with respect to slow times. The Seiberg-Witten data, including (3) and (7), can be obtained by proper parameters truncation and redefinition.

The integrable hierarchy has a gauge theory realization as “extended Seiberg-Witten theory” [14, 18, 19, 23, 25], by turning on higher Casimirs in the action,

$$\mathcal{L} = \mathcal{L}_0 + \sum_{m>1}^{\infty} t_m \text{Tr} \Phi^m, \quad (8)$$

with nilpotent couplings t_m , Φ is the $\mathcal{N} = 2$ vector superfield. Note that t_2 is actually a shift of complex gauge coupling τ_0 because the undeformed Lagrangian is $\mathcal{L}_0 = \tau_0 \text{Tr} \Phi^2$. The effective prepotential now depend on time variables t_m , denoted as $\mathcal{F}(a, \vec{t}, \Lambda)$. There exists an explicit correspondence between the extended Seiberg-Witten theory and integrable hierarchy, if properly identify variables on two sides. The scalar a_i and a_D^i is related to the

periods $\alpha_i = \oint_{A_i} dS$ and $\alpha_D^i = \oint_{B_i} dS$ of Whitham-Toda, and the prepotential \mathcal{F} is related to the logarithm of quasiclassical τ -function of Whitham-Toda.

The Seiberg-Witten theory/Toda chain data can be recovered by identifying T_1 with Λ and setting $T_n = 0, n \geq 2$. The equation (7) comes from the fact that the prepotential $\mathcal{F}(\alpha_i, T_n)$ is a homogeneous function of degree two,

$$2\mathcal{F} = \sum_i \alpha_i \frac{\partial \mathcal{F}}{\partial \alpha_i} + \sum_n T_n \frac{\partial \mathcal{F}}{\partial T_n}. \quad (9)$$

And equation (3) comes from the above equation and the following first order derivatives equation of prepotential,

$$\frac{\partial \mathcal{F}}{\partial T_n} = \frac{N}{i\pi n} \sum_m m T_m H_{m+1, n+1}. \quad (10)$$

We have to use the fact $H_{2,2} = H_2 = u_2 = \langle \text{Tr} \varphi^2 \rangle$. The Whitham-Toda hierarchy also contains equations of higher order derivatives of prepotential with respect to slow times. Especially, explicit results of the second order derivatives have been worked out in [19] through studying the higher Casimirs deformed microscopic field theory, the equations are related to the theta functions on the spectral curve, and the contact terms are fixed. It is found that from the first and second order derivative equations we can uniquely determine the prepotential of $\mathcal{N} = 2$ gauge theory in both weakly coupled region and strongly coupled region[14, 17].

As the last piece of background introduction, we would like to mention that for theory with higher rank gauge groups, the third order derivatives of prepotential with respect to a_i satisfy the generalized Witten-Dijkgraaf-Verlinde-Verlinde(WDVV) equations [20]. This is also related to the theory of Whitham hierarchies and topological field theories.

The structure of rest part of the paper is as follows. In the second section we present the Matone's relation for gauge theory in the Ω background with $\epsilon_1 \neq 0, \epsilon_2 = 0$, which does not rely on the localization method. We show that the magnetic expansion also satisfies the Matone's relation, if nonperturbative monopole corrections are properly identified. This observation is based on our previous work, and is the main reason for the present work. In the third section we briefly review a general proof of Matone's relation using localization technique. The last section is devoted to discussion on the spectrum of periodic Toda chain in the Nekrasov-Shatashvili scheme, Matone's relation indicates a relation between the quantized energy and the scattering data of Toda particles.

2 Matone's relation in presence of Ω -deformation: a case study

Integrating multi-instanton contribution turns out to be very hard, only first few multi-instanton results have been worked out and they are in accord with Seiberg-Witten result. A remarkable development in the last decade is the direct integration of instanton contributions using the localization technique[22, 23, 24, 25]. The ADHM construction of $SU(N)$ instanton solution gives the moduli space for general k -instanton configurations $\mathcal{M}_{k,N}$ of $4kN$ dimension, which is rotated by the global symmetry $SU(N) \times SO(4)$. The maximal abelian subgroup is $U(1)^{N-1} \times U(1)^2$, generated by $T_a = \exp i \text{diag}(a_1, a_2, \dots, a_N)$ constrained by $\sum_i a_i = 0$, and by $T_{\pm} = \exp i(\frac{1}{2}\epsilon_{\pm}\sigma_3)$. $U(1)^{N-1}$ is the maximal abelian subgroup of gauge group $SU(N)$, a_i is the components of the v.e.v of the scalar field in the Coulomb phase; $U(1)^2$ is the rotation of spacetime \mathbb{R}^4 along two orthogonal planes z_1, z_2 , $\epsilon_{\pm} = \epsilon_1 \pm \epsilon_2$ are complex parameters. The idea is to use the $U(1)^{N-1} \times U(1)^2$ action of the associated equivariant bundle \mathcal{E} on $\mathcal{M}_{k,N} \times \mathbb{C}^2$ to deform the gauge theory and construct multi-instanton action that is equivalent closed form under the group action. With this choice the group action has finite number of isolated fixed points in the moduli space, we can apply the localization formula to do integrals over $\mathcal{M}_{k,N}$. This construction is equivalent to turning on a particular form of background supergravity field parametered by $\epsilon_{1,2}$, called Ω -background [21, 22, 23].

The equivalent localization reduces the integrals to summation on contribution from fixed points in the moduli space, results in the Nekrasov's instanton partition function [22], a holomorphic function depending on $a_i, \epsilon_1, \epsilon_2, \Lambda$, and masses of hypermultiplets if presented,

$$Z = \exp \frac{\mathcal{F}(a, \epsilon_1, \epsilon_2, \Lambda)}{\epsilon_1 \epsilon_2}, \quad (11)$$

where $\frac{1}{\epsilon_1 \epsilon_2}$ is the leading singularity and $\mathcal{F}(a, \epsilon_1, \epsilon_2, \Lambda)$ is a regular function of $\epsilon_{1,2}$. The Ω deformed prepotential again can be divided into perturbative part and nonperturbative part, $\mathcal{F}(a, \epsilon_1, \epsilon_2) = \mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2) + \mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2)$. The Seiberg-Witten theory is recovered in the limit $\epsilon_{1,2} \rightarrow 0$. Without causing confusion, we will use the same symbols as in the Seiberg-Witten theory to express quantities like $\mathcal{F}, a, u(a)$ of the Ω deformed theory. We have to bear in mind that these quantities are all promoted to depend on the deformation parameters $\epsilon_{1,2}$.

The Ω background is a delicate deformation of $\mathcal{N} = 2$ gauge theories. It greatly simplifies the instanton integrals, meanwhile there are evidence that this deformation preserves properties of the Seiberg-Witten theory, including electric-magnetic duality and integrability. As it seems that the deformation is just right, we expect the Matone's relation should also be preserved, too. In fact, using localization technique, the Matone's relation has been

proved in the presence of Ω background[26]. However, a careful identification of $u(a)$ and the expectation of a scalar field is needed. We will explain this in the next section, before going to that, let us do a case study for a particular case which dose not rely on localization technique .

Consider $SU(N)$ pure super Yang-Mills theory in the Ω background with $\epsilon_1 \neq 0, \epsilon_2 = 0$. According to [28], the gauge theory in this background is identical to the quantization of A_{N-1} periodic Toda chain. In [29] it was proposed that the ϵ_1 deformed partition function can be obtained from the Bohr-Sommerfeld integrals of corresponding quantum mechanical system. Along this line, in [30] we have obtained the partition function of the $SU(2)$ theory through the WKB analysis of the cosine-potential quantum mechanics. Therefore, the result we present in this section does not rely on localization method and provides an independent evidence for the Matone's relation³.

The deformation parameter ϵ_1 has mass dimension one, when viewed as a component of graviphoton field. The full prepotential $\mathcal{F}(a, \epsilon_1, \Lambda) = \mathcal{F}^{pert} + \mathcal{F}^{inst}$ has dimension two, it can be written in the form $\mathcal{F} = a^2 f(\frac{\Lambda}{a}) g(\frac{\epsilon_1}{a})$, with $f(x)$ contains logarithms for perturbative effects and polynomials for instanton corrections, and $g(x)$ contains only polynomials of even order. Therefore, eq. (7) is modified to

$$2\mathcal{F} - a \frac{\partial \mathcal{F}}{\partial a} = \Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} + \epsilon_1 \frac{\partial \mathcal{F}}{\partial \epsilon_1}. \quad (12)$$

This can be easily checked using results of [30]. The renormalization flow equation $\Lambda \partial_\Lambda \mathcal{F} = \frac{2u}{i\pi}$ is not satisfied anymore. Instead, we find

$$\Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} = \frac{2u - \frac{1}{24}\epsilon_1^2}{i\pi}. \quad (13)$$

The anomaly term $\frac{1}{24}\epsilon_1^2$ comes from the 1-loop perturbative correction $\frac{1}{24}\epsilon_1^2 \ln \frac{a}{\Lambda}$. Clearly, the structure of eq. (3) is destroyed by ϵ_1 corrections.

As the anomaly comes only from perturbative effects, we expect the instanton corrections relation eq. (6) is preserved. We show that the instanton expansion indeed satisfies the Matone's relation. Up to the first few instanton expansion and ϵ_1 expansion, we have

$$\begin{aligned} -\mathcal{F}^{inst} &= \frac{1}{2}a^{-2}q + \frac{5}{64}a^{-6}q^2 + \frac{3}{63}a^{-10}q^3 + \dots \\ &+ \frac{1}{8}\epsilon_1^2 a^{-4}q + \frac{21}{128}\epsilon_1^2 a^{-8}q^2 + \frac{55}{192}\epsilon_1^2 a^{-12}q^3 + \dots \\ &+ \frac{1}{32}\epsilon_1^4 a^{-6}q + \frac{219}{1024}\epsilon_1^4 a^{-10}q^2 + \frac{1495}{1536}\epsilon_1^4 a^{-14}q^3 + \dots \end{aligned} \quad (14)$$

³We notice in recent papers[31, 32] the ϵ_1 deformed Matone's relation is discussed. Paper [31] involves a semiclassical limit of Liouville CFT that leads to quantum integrable system of [28], the Matone's relation is obtained from differential equations satisfied by the conformal blocks with degenerate operators, theories with matter hypermultiplets are discussed. Paper[32] discusses instanton counting when $\epsilon_2 = 0$ in the spirit of[23], general operators $\text{Tr}\phi^m$ are also discussed in that formalism.

where $q = \Lambda^4$ is the instanton expansion parameter. Tree and 1-loop contributions are not included. Writing \mathcal{F}^{inst} as instanton expansion

$$-\mathcal{F}^{inst} = \sum_{k=1}^{\infty} \mathcal{F}_k q^k, \quad (15)$$

now \mathcal{F}_k should be functions of a and ϵ_1 . We get

$$\begin{aligned} \mathcal{F}_1 &= \frac{1}{2}a^{-2} + \frac{1}{8}\epsilon_1^2 a^{-4} + \frac{1}{32}\epsilon_1^4 a^{-6} + \dots \\ \mathcal{F}_2 &= \frac{5}{64}a^{-6} + \frac{21}{128}\epsilon_1^2 a^{-8} + \frac{219}{1024}\epsilon_1^4 a^{-10} + \dots \\ \mathcal{F}_3 &= \frac{3}{64}a^{-10} + \frac{55}{192}\epsilon_1^2 a^{-12} + \frac{1495}{1536}\epsilon_1^4 a^{-14} + \dots \end{aligned} \quad (16)$$

On the other hand, we also have the expectation value $u = \langle \text{Tr} \varphi^2 \rangle$ in the Seiberg-Witten theory now promoted to

$$\begin{aligned} 2u &= a^2 + \frac{1}{2}a^{-2}q + \frac{5}{32}a^{-6}q^2 + \frac{9}{64}a^{-10}q^3 + \dots \\ &\quad + \frac{1}{8}\epsilon_1^2 a^{-4}q + \frac{21}{64}\epsilon_1^2 a^{-8}q^2 + \frac{55}{64}\epsilon_1^2 a^{-12}q^3 + \dots \\ &\quad + \frac{1}{32}\epsilon_1^4 a^{-6}q + \frac{219}{512}\epsilon_1^4 a^{-10}q^2 + \frac{1495}{512}\epsilon_1^4 a^{-14}q^3 + \dots \end{aligned} \quad (17)$$

Define \mathcal{G}_k through

$$2u(a) = a^2 + \sum_{k=1}^{\infty} \mathcal{G}_k q^k. \quad (18)$$

Then we have

$$\begin{aligned} \mathcal{G}_1 &= \frac{1}{2}a^{-2} + \frac{1}{8}\epsilon_1^2 a^{-4} + \frac{1}{32}\epsilon_1^4 a^{-6} + \dots \\ \mathcal{G}_2 &= \frac{5}{32}a^{-6} + \frac{21}{64}\epsilon_1^2 a^{-8} + \frac{219}{512}\epsilon_1^4 a^{-10} + \dots \\ \mathcal{G}_3 &= \frac{3}{64}a^{-10} + \frac{55}{192}\epsilon_1^2 a^{-12} + \frac{1495}{1536}\epsilon_1^4 a^{-14} + \dots \end{aligned} \quad (19)$$

\mathcal{F}_k and \mathcal{G}_k satisfy the Matone's relation

$$\mathcal{G}_k = k\mathcal{F}_k. \quad (20)$$

The sum of instanton correction is just the density of instanton gas [23],

$$\rho = \sum_{k=1}^{\infty} \mathcal{G}_k q^k \sim \exp - \frac{8\pi^2}{g^2}. \quad (21)$$

In the weak coupling theory, $g \ll 1$, the dilute gas approximation applies. However, if the coupling is strong, we have to work in the dual magnetic theory.

In [30] we have also obtained the dual magnetic and dyonic expansions of the prepotential $\mathcal{F}_{D/T}(a_{D/T}, \epsilon_1, \Lambda)$. The magnetic and dyonic expansions are actually mirror to each other, therefore, we only discuss magnetic case. The magnetic prepotential also have a structure that can be explained as perturbative part and nonperturbative part. The nonperturbative degrees of freedom is the massive hypermultiplets of the dual magnetic theory, or the monopoles of the original electric theory.

In fact, the Seiberg-Witten part of magnetic(dyonic) expansion also satisfy equations similar to (3). Including the perturbative part, they are[30]

$$\mathcal{F}_D(a_D) = \frac{1}{i\pi} \left[\frac{\hat{a}_D^2}{2} \ln\left(-\frac{\hat{a}_D}{2}\right) + 4\hat{a}_D - \frac{3}{4}\hat{a}_D^2 + \frac{1}{16}\hat{a}_D^3 + \frac{5}{512}\hat{a}_D^4 + \frac{11}{4096}\hat{a}_D^5 + \dots \right], \quad (22)$$

$$\sigma = -2\hat{a}_D + \frac{1}{4}\hat{a}_D^2 + \frac{1}{32}\hat{a}_D^3 + \frac{5}{512}\hat{a}_D^4 + \dots. \quad (23)$$

They satisfy a dual form of equation (3), namely

$$2\mathcal{F}_D - \hat{a}_D \frac{\partial \mathcal{F}_D}{\partial \hat{a}_D} = -\frac{2\sigma}{i\pi}, \quad (24)$$

For the dyonic case, it is

$$2\mathcal{F}_T - a_T \frac{\partial \mathcal{F}_T}{\partial a_T} = \frac{2\varpi}{i\pi}. \quad (25)$$

with $\varpi = u + \Lambda^2$. The left hand side of equations (24) and (25) can be written as renormalization equations, too.

Due to the ϵ_1 corrections, equations (24) and (25) are not satisfied by the dual deformed prepotential $\mathcal{F}_D(a_D, \epsilon_1, \Lambda)$. However, if we properly identify nonperturbative corrections, the Matone's relation still holds. The nonperturbative part we identify is

$$\begin{aligned} \mathcal{F}_D^{monop} &= \frac{1}{16}\hat{a}_D^3 q_D + \frac{5}{512}\hat{a}_D^4 q_D^2 + \frac{11}{4096}\hat{a}_D^5 q_D^3 + \dots \\ &+ \frac{\epsilon_1^2}{2^5} \left(-\frac{3}{2^3}\hat{a}_D q_D - \frac{17}{2^7}\hat{a}_D^2 q_D^2 - \frac{205}{2^{10} \times 3}\hat{a}_D^3 q_D^3 + \dots \right) \\ &+ \frac{\epsilon_1^4}{2^{11}} \left(\frac{135}{2^9}\hat{a}_D q_D^3 + \frac{2943}{2^{13}}\hat{a}_D^2 q_D^4 + \dots \right). \end{aligned} \quad (26)$$

We have introduced the monopole expansion parameter $q_D = \frac{1}{\Lambda}$. Again, perturbative contributions are not included. Define the monopole expansion as

$$\mathcal{F}_D^{monop} = \sum_{k=1}^{\infty} \mathcal{F}_k^D q_D^k, \quad (27)$$

then we have

$$\begin{aligned} \mathcal{F}_1^D &= \frac{1}{16}\hat{a}_D^3 - \frac{3}{2^8}\epsilon_1^2 \hat{a}_D, \\ \mathcal{F}_2^D &= \frac{5}{512}\hat{a}_D^4 - \frac{17}{2^{12}}\epsilon_1^2 \hat{a}_D^2, \\ \mathcal{F}_3^D &= \frac{11}{4096}\hat{a}_D^5 - \frac{205}{2^{15} \times 3}\epsilon_1^2 \hat{a}_D^3 + \frac{135}{2^{20}}\epsilon_1^4 \hat{a}_D. \end{aligned} \quad (28)$$

Note that these polynomials have finite terms, because of the “gap” structure of the dual expansion of prepotential.

The proper local moduli coordinate near $u = \Lambda^2$ is $\sigma = u - \Lambda^2$, we have

$$\begin{aligned}\sigma = & -2\hat{a}_D q_D^{-1} + \frac{1}{4}\hat{a}_D^2 + \frac{1}{32}\hat{a}_D^3 q_D + \frac{5}{512}\hat{a}_D^4 q_D^2 + \frac{33}{8192}\hat{a}_D^5 q_D^3 + \cdots \\ & + \frac{\epsilon_1^2}{2^6}(-1 - \frac{3}{8}\hat{a}_D q_D - \frac{17}{64}\hat{a}_D^2 q_D^2 - \frac{205}{1024}\hat{a}_D^3 q_D^3 + \cdots) \\ & + \frac{\epsilon_1^4}{2^{17}}(9q_D^2 + \frac{405}{16}\hat{a}_D q_D^3 + \frac{2943}{64}\hat{a}_D^2 q_D^4 + \frac{69001}{1024}\hat{a}_D^3 q_D^5 + \cdots).\end{aligned}\quad (29)$$

Define \mathcal{G}_k^D through

$$2\sigma = < 2\sigma >_0 + \sum_{k=1}^{\infty} \mathcal{G}_k^D q_D^k, \quad (30)$$

where $< 2\sigma >_0$ represents terms that cannot be interpreted as monopole corrections. For example, the first monopole correction should starts from terms of order q_D . Also, the first term in ϵ_1^4 expansion seems not a monopole contribution, actually all the first terms in higher ϵ_1 expansions should be excluded. The validity of the dual Matone’s relation presented in the following indicates validity of this choice. We do not face this problem for electric expansions since it is straightforward to recognize all instanton terms. We have

$$\begin{aligned}\mathcal{G}_1^D &= \frac{1}{16}\hat{a}_D^3 - \frac{3}{2^8}\epsilon_1^2 \hat{a}_D, \\ \mathcal{G}_2^D &= \frac{5}{256}\hat{a}_D^4 - \frac{17}{2^{11}}\epsilon_1^2 \hat{a}_D^2, \\ \mathcal{G}_3^D &= \frac{33}{4096}\hat{a}_D^5 - \frac{205}{2^{15}}\epsilon_1^2 \hat{a}_D^3 + \frac{405}{2^{20}}\epsilon_1^4 \hat{a}_D.\end{aligned}\quad (31)$$

\mathcal{F}_k^D and \mathcal{G}_k^D have finite terms, they also satisfy the Matone’s relation

$$\mathcal{G}_k^D = k\mathcal{F}_k^D. \quad (32)$$

The sum of monopole corrections is

$$\rho_D = \sum_{k=1}^{\infty} \mathcal{G}_k^D q_D^k \sim \exp - \frac{4\pi^2}{g_D^2}, \quad (33)$$

when the dual coupling g_D is small, this can be interpreted as density of dilute monopole gas.

The existence of Matone’s relation in the magnetic theory can be viewed as the consequence of compatibility of integrability and duality. In fact, the associated integrable hierarchies are closely related to modular forms on the spectral curve, as the τ function of algebraic integrable system is essentially the Riemann theta function. Explicit results of

the genus zero and genus one gravitational contributions to the prepotential $\mathcal{F}(a, \epsilon_1, \epsilon_2, \Lambda)$ presented in [25] indicate their dependence on modular forms.

A bonus of the Matone's relation is application to the theory of the Mathieu differential equation. We have analysed the relation between SU(2) Seiberg-Witten gauge theory and the Mathieu equation in [30]. The eigenvalue formulae (24) and (31) in the second paper of [30] are precisely the instanton expansions, in the terminology of gauge theory, if we notice the correspondence $\lambda_\nu = 8\epsilon_1^{-2}u$, $\nu = 2\epsilon_1^{-1}a$, $q = 4\epsilon_1^{-2}\Lambda^2$ in that paper (Attention: these are the symbols we have used in our previous paper [30]. The expansion parameter q is not the same $q = \Lambda^4$ in this paper, this notation switch applies only when we refer to the Mathieu equation). In fact, every coefficient of q^{2k} of the Mathieu eigenvalue expansion λ_ν in paper [30] is precisely $4\mathcal{G}_k$ of $\text{Tr}\varphi^2$ of gauge theory. Indeed, it is easy to check that the coefficients \mathcal{F}_k obtained by instanton counting, after setting $\epsilon_2 = 0$ and multiplying $4k$, exactly coincide with formulae (24) and (31). For example, the first paper of [27] gives explicit results of \mathcal{F}_k up to 4-instantons convenient for our comparison, the first 3-instanton corrections are

$$\mathcal{F}^{inst}|_{\epsilon_2=0} = \frac{2}{4a^2 - \epsilon_1^2}\Lambda^4 + \frac{20a^2 + 7\epsilon_1^2}{(4a^2 - \epsilon_1^2)^3(4a^2 - 4\epsilon_1^2)}\Lambda^8 + \frac{16(144a^4 + 29\epsilon_1^4 + 232a^2\epsilon_1^2)}{3(4a^2 - \epsilon_1^2)^5(4a^2 - 4\epsilon_1^2)(4a^2 - 9\epsilon_1^2)}\Lambda^{12} + \dots \quad (34)$$

While the first 3-terms q^2 -expansion of the Mathieu eigenvalue λ_ν are

$$\lambda_\nu - \nu^2 = \frac{1}{2(\nu^2 - 1)}q^2 + \frac{5\nu^2 + 7}{32(\nu^2 - 1)^3(\nu^2 - 4)}q^4 + \frac{9\nu^4 + 58\nu^2 + 29}{64(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)}q^6 + \dots \quad (35)$$

Precisely we have $4k\mathcal{F}_k|_{\epsilon_2=0} = \lambda_\nu^{(k)}$, as expected. Therefore, along the line of [22] by counting all possible partitions $k = k_1 + k_2$ with the corresponding 2-Young diagrams $\{Y_1, Y_2\}$, we can evaluate \mathcal{F}_k , thus get \mathcal{G}_k via the Matone's relation, which essentially is the coefficient of q^{2k} of the Mathieu eigenvalue expansion for $\frac{q}{\nu^2} \ll 1$.

However, the story of the magnetic theory is less well understood, compare to the electric theory, although the dual relation is observed. Notice that the formula (34) in the second paper of [30] is

$$\lambda_\nu = 2q - 4\nu\sqrt{q} + \frac{4\nu^2 - 1}{2^3} + \frac{4\nu^3 - 3\nu}{2^6\sqrt{q}} + \frac{80\nu^4 - 136\nu^2 + 9}{2^{12}q} + \frac{528\nu^5 - 1640\nu^3 + 405\nu}{2^{16}q^{\frac{3}{2}}} + \dots \quad (36)$$

Terms of order $q^{\frac{k}{2}}$ are precisely monopole corrections, but few leading terms are not. Setting $\nu = 0$ we get the ground state energy $\langle \sigma \rangle_0$, this includes terms such as $9/(2^{12}q)$, which have the form of monopole corrections but are incompatible with the dual prepotential expansion and dual Matone's relation, thus excluded from \mathcal{G}_k^D . As we do not have an explicit algorithm to counting monopole correction \mathcal{F}_k^D , thus there is no Young-diagram-counting method to evaluate the Mathieu eigenvalue expansion for $\frac{q}{\nu^2} \gg 1$.

3 Matone's relation from localization

In [26] it was proved that the Matone's relation (6) should hold for general ϵ_1, ϵ_2 , based on localization technique. We briefly review the results here for completeness. We need to know instanton corrections to $\mathcal{F}(a)$ and $u(a)$ of the Ω deformed theory. The prepotential can be extracted from the Nekrasov's partition function. The integral of k -instanton contribution localized to the fixed points of the $U(1)^{N-1} \times U(1)^2$ action. These fixed points are in one to one correspondence with the partitions of N integers k_α satisfying $k = k_1 + k_2 + \cdots + k_N$. Each k_α is accompanied with a Young diagram Y_α with k_α boxes, i.e. $|Y_\alpha| = k_\alpha$, therefore for each partition we have a group of N-Young diagrams $\{Y_\alpha, \alpha = 1, 2, \cdots N\}$. For each box s in the i_α -th row and j_α -th column of Y_α , define the following counting data:

$$h_\beta(s) = \nu_{\beta, i_\alpha} - j_\alpha, \quad v_\beta(s) = \bar{\nu}_{\beta, j_\alpha} - i_\alpha. \quad (37)$$

where ν_{β, i_α} is the number of boxes in the i_α -th row of Y_β , and $\bar{\nu}_{\beta, j_\alpha}$ is the number of boxes in the j_α -th column of Y_β . Indexes α and β are not necessary the same, therefore when they are different the box $s(i_\alpha, j_\alpha)$ may be located outside the Young diagram Y_β . The quantity $h_\beta(s)$ is the horizontal distance from the position of the box s till the right end of the Y_α diagram, and $v_\alpha(s)$ is the vertical distance from the position of the box s till the upper end of the Y_β diagram.

Then the result of k -instanton contribution can be written in the form [27]

$$Z_k = \sum_{\{Y_\alpha, \sum |Y_\alpha| = k\}} \prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} \frac{1}{E_{\alpha\beta}(s)(\epsilon_+ - E_{\beta\alpha}(s'))}, \quad (38)$$

the sum is running over all possible partitions, and

$$E_{\alpha\beta}(s) = a_{\alpha\beta} - h_\beta(s)\epsilon_1 + (v_\alpha(s) + 1)\epsilon_2, \quad (39)$$

where $a_{\alpha\beta} = a_\alpha - a_\beta$. The full instanton partition function is

$$Z^{inst} = \sum_{k=1}^{\infty} Z_k q^k. \quad (40)$$

For $u(a)$, we have to properly deform $\langle \text{Tr} \varphi^2 \rangle$ in the Seiberg-Witten theory to suit the Ω background. In order to use localization, we have to construct an integral $\langle \text{Tr} \tilde{\varphi}^2 \rangle$ that is equivalent closed form under the equivalent differential. Moreover, the integral must recover the corresponding integral of the Seiberg-Witten $\langle \text{Tr} \varphi^2 \rangle$ when taking the limit $\epsilon_{1,2} \rightarrow 0$. It turns out that the proper deformed scalar field is

$$\tilde{\varphi} = \tilde{\varphi}_{bos} + \varphi_{ferm}, \quad (41)$$

with

$$\tilde{\varphi}_{bos} = \bar{U}(z)i \begin{pmatrix} a & 0 \\ 0 & \phi + \frac{1}{2}\epsilon_- \sigma_3 \end{pmatrix} U(z) - \bar{U}(z)iz_l \epsilon_l \frac{\partial}{\partial z_l} U(z), \quad (42)$$

where $U(z)$ comes from the self-dual gauge field $A_\mu = \bar{U}(x)\partial_\mu U(x)$. ϕ comes from $T_\phi = \exp i\phi \in U(k)$ which is the redundant symmetry of the ADHM constraints, the value of its diagonal components ϕ_s at a fixed point are determined by the k -boxes of the corresponding Young diagrams $\{Y_\alpha\}$,

$$\phi_{(i_\alpha, j_\alpha)} = a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2. \quad (43)$$

φ_{ferm} is the solution of the scalar field when no v.e.v. and ϵ deformation presented, constructed from ADHM data, involving a fermion bilinear therefore the subscript.

Actually, using the localization technique a general formula for chiral ring operators $\langle \text{Tr} \tilde{\varphi}^m \rangle$ can be obtained, it is [18, 25, 26]

$$\langle \text{Tr} \tilde{\varphi}^m \rangle = \frac{1}{Z^{inst}} \sum_{k=1}^{\infty} \sum_{\{Y_\alpha, \sum |Y_\alpha| = k\}} \frac{\mathcal{O}_m(\{Y_\alpha\})}{\prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} E_{\alpha\beta}(s)(\epsilon_+ - E_{\beta\alpha}(s'))} q^k. \quad (44)$$

with

$$\begin{aligned} \mathcal{O}_m(\{Y_\alpha\}) &= \sum_{\alpha=1}^N \{a_\alpha^m - \sum_{s(i_\alpha, j_\alpha) \in Y_\alpha} [(a_\alpha + j_\alpha \epsilon_1 + i_\alpha \epsilon_2)^m - (a_\alpha + j_\alpha \epsilon_1 + (i_\alpha - 1)\epsilon_2)^m \\ &\quad - (a_\alpha + (j_\alpha - 1)\epsilon_1 + i_\alpha \epsilon_2)^m + (a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2)^m]\}. \end{aligned} \quad (45)$$

can be read from the Chern character of the equivariant bundle \mathcal{E} over a fixed point,

$$Ch_{\bar{Y}}(\mathcal{E}) = \sum_{\alpha}^N [e^{ia_\alpha} - (1 - e^{i\epsilon_1})(1 - e^{i\epsilon_2}) \sum_{s(i_\alpha, j_\alpha) \in Y_\alpha} e^{i[a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2]}]. \quad (46)$$

For the case of $m = 2$, the operator $\mathcal{O}_2(\{Y_\alpha\})$ greatly simplifies and does not depend on the shape of Young diagram,

$$\begin{aligned} \mathcal{O}_2(\{Y_\alpha\}) &= \sum_{\alpha=1}^N (a_\alpha^2 - 2k_\alpha \epsilon_1 \epsilon_2) \\ &= a_1^2 + a_2^2 + \cdots + a_N^2 - 2k \epsilon_1 \epsilon_2. \end{aligned} \quad (47)$$

Therefore we have

$$\begin{aligned}
\langle \text{Tr} \tilde{\varphi}^2 \rangle &= \frac{1}{\sum_k Z_k q^k} \sum_k \sum_{\{Y_\alpha\}} \prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} \frac{(\sum_\gamma a_\gamma^2 - 2k_\gamma \epsilon_1 \epsilon_2) q^k}{E_{\alpha\beta}(s)(\epsilon_+ - E_{\beta\alpha}(s'))} \\
&= \sum_\gamma a_\gamma^2 - 2\epsilon_1 \epsilon_2 \frac{\sum_k Z_k k q^k}{Z^{inst}} \\
&= \sum_\gamma a_\gamma^2 - 2\epsilon_1 \epsilon_2 q \frac{\partial}{\partial q} \ln Z^{inst} \\
&= a_1^2 + a_2^2 + \cdots + a_N^2 + \sum_{k=1}^{\infty} 2k \mathcal{F}_k q^k. \tag{48}
\end{aligned}$$

We have used $\epsilon_1 \epsilon_2 \ln Z^{inst} = \mathcal{F}^{inst} = -\sum_k \mathcal{F}_k q^k$. Considering $u(a, \epsilon_1, \epsilon_2, \Lambda) = \langle \text{Tr} \tilde{\varphi}^2 \rangle$ can also be expanded as $u(a) = a_1^2 + a_2^2 + \cdots + a_N^2 + \sum_{k=1}^{\infty} 2\mathcal{G}_k q^k$ by definition, then the Matone's relation $\mathcal{G}_k = k\mathcal{F}_k$ immediately follows.

4 Implication for energy spectrum of Toda chain

According to Nekrasov-Shatashvili scheme[28], a large class of four dimensional $\mathcal{N} = 2$ super Yang-Mills theories deformed by Ω background with $\epsilon_1 \neq 0, \epsilon_2 = 0$ are identical to quantization of the corresponding classical integrable systems [11, 12]. The deformation parameter ϵ_1 plays the role of the Plank constant.

The integrable model/ $\mathcal{N} = 2$ theory correspondence establishes a relation between the periodic Toda chain and $\mathcal{N} = 2$ pure super Yang-Mills. The periodic Toda chain of length L is defined as, in a dimensionless form,

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + e^{-\frac{L}{N}} \sum_{i=1}^N e^{x_i - x_{i+1}}, \tag{49}$$

with periodic boundary condition $x_{N+1} = x_1$. Its stationary region is at $x_1 = x_2 = \cdots = x_N$. The open Toda chain can be obtained by cutting the closed chain, i.e., eliminating the interaction term $\exp(x_N - x_1)$, its asymptotic region is at $x_i - x_{i+1} \rightarrow -\infty$.

Gutzwiller studied the quantization condition for periodic Toda chain in [33], based on the work of Kac and van Moerbeke on classical chain [34]. Kac and van Moerbeke constructed a new set of conjugate coordinates variables for N-particle periodic chain from the asymptotic momenta of N-1 particle open chain obtained by cutting the closed chain. This is a nonlinear canonical transformation, the resulting mechanical system gives the same conserved quantities as the original one, but have simple evolution behavior: its equation of motion is separable and breaks up into N-1 equations depending on only one variable. Gutzwiller's method

is the quantum analogy of the classical transformation. The wave function of quantum N-particle periodic chain $\Psi_N(x_1, x_2, \dots, x_N)$ is constructed as linear combination of asymptotic wave function of N-1 particle open Toda chain $\psi_{N-1}(x_1, x_2, \dots, x_{N-1}; \{k_i\})$. The wave numbers k_i are zeros of the Hill-type determinant. In the case of $N = 2$, this construction coincides with theory of modified Mathieu differential equation. The vanishing of the wave function for large separation $\exp(x_i - x_{i+1}) \gg 1$ requires k_i satisfy the Gutzwiller's quantization conditions. Essentially Gutzwiller's solution of periodic Toda chain spectrum is in the realm of Sklyanin's separation of variables[35]. However, the resulting quantization conditions involve both zeros of Hill's determinant and zeros of the characteristic polynomial of Lax matrix, it is not easy to handle in practice.

The Nekrasov-Shatashvili scheme provides a novel way to quantize these integrable systems. The main point is that the prepotential of the gauge theory is identified with the Yang-Yang function[36] of the integrable system, a_i is identified with the momentum of the i -th (quasi)particle, and the instanton corrections are identified with finite size corrections $\Lambda^{2N} \sim e^{-L}$. The quantization condition is given by the Bethe equation,

$$\frac{\partial \mathcal{F}(a, \epsilon_1, \Lambda)}{\partial a_i} = n_i, \quad n_i \in \mathbb{Z}. \quad (50)$$

As demonstrated in [37], the Bethe equation (50) is consistent with Gutzwiller's quantization of periodic Toda chain. The eigenvalues of quantum Hamiltonians can be obtained from the expectation values of chiral ring operators $\text{Tr} \tilde{\varphi}^m$. Deform the gauge theory action as

$$\mathcal{L}_{\epsilon, t} = \mathcal{L}_{\epsilon} + \sum_{m>1} t_m \text{Tr} \tilde{\varphi}^m, \quad (51)$$

where \mathcal{L}_{ϵ} is the Ω deformed action. The deformed action is still an equivalent closed form and the localization can be applied, result in a prepotential $\mathcal{F}(a, \epsilon_1, \epsilon_2, \vec{t}, \Lambda)|_{\epsilon_2 \rightarrow 0}$. Then we have

$$\mathcal{E}_m = \frac{\partial \mathcal{F}(a, \epsilon_1, \vec{t}, \Lambda)}{\partial t_m} \Big|_{\vec{t}=0}. \quad (52)$$

Therefore, using the explicit formulae presented in [18, 25, 26], the quantum Hamiltonian of periodic Toda chain can be evaluated, at least in the electric frame. Especially, \mathcal{E}_2 is the energy of the Toda chain. The Matone's relation discussed in the last two sections tells us that \mathcal{E}_2 can be directly read from the Yang-Yang function,

$$\mathcal{E}_2 = a_1^2 + a_2^2 + \dots + a_N^2 + \sum_{k=1}^{\infty} 2k \mathcal{F}_k q^k \Big|_{\epsilon_2=0}. \quad (53)$$

This is in accordance with (6.13) of [28]. The momenta a_i have to obey the condition $a_1 + a_2 + \dots + a_N = 0$ and the Bethe quantization condition (50).

In the discussion above, a_i are the momenta of the quasi particles. In interacting multi particle systems quasiparticles are the proper degrees of freedom(d.o.f) to describe the dynamics of excitations. For example, the effective excitation d.o.f of spin system are magnons, and the effective excitation d.o.f of crystal lattice are phonons. When a_i satisfy the large momenta condition $a_i \gg \Lambda, a_i - a_j \gg \Lambda$, the gauge theory is weakly coupled. It turns out that the properties of the excitations of periodic Toda chain change when the momenta become small. This is related to the electric-magnetic duality of the corresponding gauge theory.

From the point of gauge theory this is quite reasonable, as the electric-magnetic duality is main tool to understand dynamics of gauge theories [38]. The spirit is that at different part in the moduli space, there exists different d.o.f that are convenient for the formulation of the theory. In the weakly coupled region, the proper d.o.f are the weakly coupled electric fields; in the strongly coupled region, the proper d.o.f are the dual magnetic fields. In the gauge theory/integrable system correspondence, the moduli space of gauge theory is isomorphic to the space of Hamiltonians of (complexified) integrable system. Therefore, translate the logic of gauge theory duality to the corresponding integrable system, it indicates the quasiparticle d.o.f may change when we move from one region to another region in the space of Hamiltonians $\{\mathcal{E}_m\}$. For the case of Toda chain, if the momenta, therefor the Hamiltonians, are all large, the proper d.o.f are the quasi particles with momenta a_i . The leading order dispersion relation is nonrelativistic, $\mathcal{E}_2 \sim a^2$. These short wave length particles feel finite size corrections scale as $\delta\mathcal{E}_2 \sim (\frac{\Lambda}{a})^{2Nk}$. However, at certain point corresponding to the strongly coupled gauge theory, the proper d.o.f should be some other kind quasiparticles with small momenta denoted as a_D , satisfying $a_D \ll \Lambda$. For example, at the point of the maximal strong coupling singularities where the maximal number of mutually local particles become massless, the excitations of Toda chain are all magnetic quasiparticles with momenta a_{Di} . This happens when the maximal number of distinct pairs of branch points of the Seiberg-Witten curve(spectral curve) collide, i.e. the curve $y^2 = P_N^2(x) - \Lambda^{2N} = \prod_{i=1}^{2N}(x - e_i(u_k))$ has only double zeros. Each colliding pair of branch points corresponds to a shrinking homology cycle, as the intersection number of these shrinking homology cycles are zero, the corresponding massless particles are mutually local. We have($u = u_2$) [39]

$$u - u_{max} = -2\Lambda \sum_{i=1}^{N-1} \hat{a}_{Di} \sin \frac{i\pi}{N} + \frac{1}{4} \sum_{i=1}^{N-1} \hat{a}_{Di}^2 + \mathcal{O}(\frac{\hat{a}_{Di}^3}{\Lambda}). \quad (54)$$

Other higher Casimirs $u_m = \langle \text{Tr} \varphi^m \rangle$ also have linear leading order dispersion relations(see (19) of [17]). Therefor, for quasiparticles of this kind the leading order dispersion relation is relativistic, $\mathcal{E}_2 \sim a_D$. These long wave length quasiparticles feel finite size corrections scale as $\delta\mathcal{E}_2 \sim (\frac{a_D}{\Lambda})^k$.

In order to have a full understanding about the quantum spectrum of small momenta excitations, we have to work out the modular form of the Ω deformed extended prepotential $\mathcal{F}(a, \epsilon_1, \vec{t}, \Lambda)$ for general gauge groups, this is still a challenging problem.

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References

- [1] N. Seiberg, E. Witten, “Electric-Magnetic Duality, Monopole Condensation, And Confinement In $N = 2$ Supersymmetric Yang-Mills Theory”, Nucl.Phys. **B426**, 19-52(1994), Erratum-ibid. **B430**, 485-486(1994), [hep-th/9407087];
 “Monopoles, Duality and Chiral Symmetry Breaking in $N=2$ Supersymmetric QCD”, Nucl.Phys. **B431**, 484-550(1994), [hep-th/9408099].
- [2] M. Matone, “Instantons and recursion relations in $N=2$ Susy gauge theory”, Phys. Lett. **B357** (1995) 342, [hep-th/9506102].
- [3] J. Sonnenschein, S. Theisen, S. Yankielowicz, “On the Relation Between the Holomorphic Prepotential and the Quantum Moduli in SUSY Gauge Theories”, Phys.Lett. **B367** (1996) 145-150, [hep-th/9510129].
 T. Eguchi, S-K. Yang, “Prepotentials of $N=2$ Supersymmetric Gauge Theories and Soliton Equations”, Mod.Phys.Lett. **A11** (1996) 131-138, [hep-th/9510183].
- [4] F. Fucito, G. Travaglini, “Instanton Calculus and Nonperturbative Relations in $N=2$ Supersymmetric Gauge Theories”, Phys.Rev. **D55** (1997) 1099-1104, [hep-th/9605215].
- [5] N. Dorey, V.V. Khoze, M.P. Mattis, “Multi-Instanton Check of the Relation Between the Prepotential F and the Modulus u in $N=2$ SUSY Yang-Mills Theory”, Phys.Lett. **B390** (1997) 205-209, [hep-th/9606199].
- [6] P. S. Howe, P. C. West, “Superconformal Ward Identities and $N=2$ Yang-Mills Theory”, Nucl.Phys. **B** 486 (1997) 425-442, [hep-th/9607239].
- [7] G. Bonelli, M. Matone, “Nonperturbative Renormalization Group Equation and Beta Function in $N=2$ SUSY Yang-Mills”, Phys.Rev.Lett. **76** (1996) 4107-4110, [hep-th/9602174].

- [8] Eric D'Hoker, I.M. Krichever, and D.H. Phong, "The Renormalization Group Equation in N=2 Supersymmetric Gauge Theories", Nucl.Phys. B494 (1997) 89-104, [hep-th/9610156].
- [9] Eric D'Hoker, D.H. Phong, "Calogero-Moser Systems in SU(N) Seiberg-Witten Theory", Nucl.Phys. B513 (1998) 405-444, [hep-th/9709053].
- [10] J. D. Edelstein, M. Gomez-Reino, J. Mas, "Instanton corrections in N=2 supersymmetric theories with classical gauge groups and fundamental matter hypermultiplets", Nucl.Phys. B561 (1999) 273-292, [hep-th/9904087].
- [11] A. Gorsky, I.M. Krichever, A. Marshakov, A. Mironov and A. Morozov, "Integrability and Seiberg-Witten exact solution" Phys. Lett. B355(1995)466 [hep-th/9505035].
E. Martinec, N. Warner, "Integrable systems and supersymmetric gauge theories", Nucl. Phys. B459 (1996) 97-112, [hep-th/9509161].
- [12] R. Donagi, E. Witten, "Supersymmetric Yang-Mills and integrable systems", Nucl.Phys. B460(1996)299-334, [hep-th/9510101].
- [13] T. Nakatsu, K. Takasaki, "Whitham-Toda hierarchy and N = 2 supersymmetric Yang-Mills theory", Mod.Phys.Lett. A11 (1996) 157-168, [hep-th/9509162].
H. Itoyama, A. Morozov, "Prepotential and the Seiberg-Witten Theory", Nucl.Phys. B491 (1997) 529-573, [hep-th/9512161]
- [14] A. Gorsky, A. Marshakov, A. Mironov, A. Morozov, "RG Equations from Whitham Hierarchy", Nucl.Phys.B527(1998)690-716, [hep-th/9802007].
J. D. Edelstein, M. Marino, J. Mas, "Whitham Hierarchies, Instanton Corrections and Soft Supersymmetry Breaking in N=2 SU(N) Super Yang-Mills Theory", Nucl.Phys. B541 (1999) 671-697, [hep-th/9805172].
- [15] I. M. Krichever, "The τ -function of the universal Whitham hierarchy, matrix models and topological field theories", Commun. Pure Appl. Math.47(1994)437, [hep-th/9205110].
- [16] O. Babelon, D. Bernard, M. Talon, "*Introduction to Classical Integrable Systems*", Chapter 10, Cambridge University Press, Cambridge, 2003.
- [17] J. D. Edelstein, J. Mas, "Strong coupling expansion and Seiberg-Witten-Whitham equations", Phys.Lett. B452 (1999) 69-75, [hep-th/9901006].
- [18] A. Losev, A. Marshakov, N. Nekrasov, "Small instantons, little strings and free fermions", In "From fields to strings", vol. 1, 581-621, [hep-th/0302191].

- [19] A. Marshakov, N. Nekrasov, “Extended Seiberg-Witten Theory and Integrable Hierarchy”, JHEP 0701(2007)104, [hep-th/0612019].
- [20] G. Bonelli, M. Matone, “Nonperturbative Relations in N=2 SUSY Yang-Mills and WDVV equation”, Phys.Rev.Lett. 77 (1996) 4712-4715, [hep-th/9605090].
 K. Ito, S-K. Yang, “The WDVV Equations in N=2 Supersymmetric Yang-Mills Theory”, Phys.Lett. **B433** (1998) 56-62, [hep-th/9803126].
 A.Marshakov, A.Mironov, A.Morozov, “WDVV-like equations in N=2 SUSY Yang-Mills Theory”, Phys.Lett. **B389** (1996)43-52, [hep-th/9607109]. “More Evidence for the WDVV Equations in N=2 SUSY Yang-Mills Theories”, Int.J.Mod.Phys. **A15** (2000) 1157-1206, [hep-th/9701123].
- [21] A. Losev, N. Nekrasov, S. L. Shatashvili, “Issues in Topological Gauge Theory”, Nucl. Phys. **B 534** (1998) 549, [hep-th/9711108].
 G. Moore, N. Nekrasov, S. L. Shatashvili, “Integrating Over Higgs Branches”, Commun.Math.Phys. 209 (2000) 97-121, [hep-th/9712241].
- [22] N. Nekrasov, “Seiberg-Witten Prepotential From Instanton Counting”, Adv. Theor. Math. Phys. **7**, 831-864(2004), [hep-th/0206161]. Also, Proceedings of the ICM, Beijing 2002, vol. **3**, 477–496, [hep-th/0306211]
- [23] N. Nekrasov and A. Okounkov, “Seiberg-Witten Theory and Random Partitions”, [hep-th/0306238].
- [24] H. Nakajima, K. Yoshioka, “Instanton counting on blowup. I. 4-dimensional pure gauge theory”, Inventiones Mathematicae, Vol.162, Number **2**, 313-355, [math/0306198].
- [25] H. Nakajima, K. Yoshioka, “Lectures on Instanton Counting”, Proceedings of “Workshop on algebraic structures and moduli spaces”, July 2003, [math/0311058].
- [26] R. Flume, F. Fucito, J. F. Morales, R. Poghossian, “Matone’s Relation in the Presence of Gravitational Couplings”, JHEP 0404 (2004) 008, [hep-th/0403057].
- [27] R. Flume and R. Poghossian, “An Algorithm for the Microscopic Evaluation of the Coefficients of the Seiberg-Witten Prepotential”, Int. J. Mod. Phys. **A 18** (2003) 2541, [hep-th/0208176].
 U. Bruzzo, F. Fucito, J. F. Morales, A. Tanzini, “Multi-Instanton Calculus and Equivariant Cohomology”, JHEP 0305(2003)054, [hep-th/0211108].

- [28] N. Nekrasov, S. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories”, 16th International Congress on Mathematical Physics, Prague, August 2009, P. Exner, Editor, pp.265-289, World Scientific 2010, [arXiv:0908.4052].
- [29] A. Mironov, A. Morozov, “Nekrasov Functions and Exact Bohr-Sommerfeld Integrals”, JHEP. **04**(2010)040, [arXiv:0910.5670 [hep-th]]. “Nekrasov Functions from Exact BS Periods: the Case of $SU(N)$ ”, J.Phys. **A43**(2010)195401, [arXiv:0911.2396[hep-th]]. A. Popolitov, “On relation between Nekrasov functions and BS periods in pure $SU(N)$ case”, [arXiv:1001.1407[hep-th]].
- [30] W. He, Y-G. Miao, “Magnetic expansion of Nekrasov theory: the $SU(2)$ pure gauge theory”, Phys. Rev. **D 82** (2010)025020, [arXiv:1006.1214 [hep-th]]. “Mathieu equation and Elliptic curve”, [arXiv:1006.5185[math-ph]].
- [31] K. Maruyoshi, M. Taki, “Deformed Prepotential, Quantum Integrable System and Liouville Field Theory”, Nucl.Phys. **B841**(2010)388-425, [arXiv:1006.4505 [hep-th]].
- [32] R. Poghossian, “Deforming SW curve”, [arXiv:1006.4822 [hep-th]].
- [33] M. C. Gutzwiller, “The quantum mechanical Toda lattice”, Ann. Phys. **124**(1980)347; “The quantum mechanical Toda lattice II”, Ann. Phys.**133**(1981)304.
- [34] M. Kac, P. van Moerbeke, “On Some Periodic Toda Lattices”, Proc. Nat. Acad. Sci. USA **72**, No. 4(1975)1627-1629; “A complete solution of the periodic Toda problem”, Proc. Nat. Acad. Sci. USA **72**, No. 8(1975)2879-2880.
P. van Moerbeke, “The Spectrum of Jacobi Matrices”, Inventiones math. **37**(1976)45-81.
- [35] E. K. Sklyanin, “The quantum Toda chain”, Lecture Notes in Physics, vol. **226**, Springer, Berlin, 1985; “Separation of Variables. New Trends.”, Prog. Theor. Phys. Suppl. 118(1995)35-60, [arXiv:solv-int/9504001].
- [36] C. N. Yang, C. P. Yang, “Thermodynamics of 1-d system of Bosons with repulsive delta function interaction”, J. Math. Phys. **10** (1969), 1115.
- [37] K. K. Kozłowski, J. Teschner, “TBA for the Toda chain”, [arXiv:1006.2906 [math-ph]].
- [38] N. Seiberg, “Exact Results on the Space of Vacua of Four Dimensional SUSY Gauge Theories”, Phys.Rev. **D49**(1994)6857-6863, [hep-th/9402044].
K. Intriligator, N. Seiberg, “Lectures on supersymmetric gauge theories and electric-magnetic duality”, Nucl.Phys.Proc.Suppl.**45BC**(1996)1-28, [hep-th/9509066].

- [39] M. R. Douglas and S. H. Shenker, “Dynamics of $SU(N)$ supersymmetric gauge theory”, Nucl. Phys. **B447**(1995)271-296, [hep-th/9503163]. E. D’Hoker, D.H.Phong, “Strong coupling expansions of $SU(N)$ Seiberg-Witten theory”, Phys.Lett.**B397**(1997)94-103,[hep-th/9701055].